ON THE DISPLACEMENT OF AN ABSOLUTELY RIGID BODY UNDER THE ACTION OF AN ACOUSTIC PRESSURE WAVE

(O PEREMESHCHENII ABSOLIUTNO TVERDOGO TELA POD DEISTVIEM AKUSTICHESKOI VOLNY DAVLENIIA)

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V.V. NOVOZHILOV
(Leningrad)

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Let us study an absolutely rigid body, free of supports, that is immersed in an infinite liquid. Assume that in the medium surrounding the body a pressure wave, whose potential is $\Phi(x - ct) = \Phi(\xi)$, is propagated. At time $t = 0$ its front comes into contact with the surface of the initially stationary body. Let us assume that the function $\Phi(\xi)$ tends to some limit as $\xi \to -\infty$. The latter means that the total pressure impulse of the wave

$$I = \int_0^\infty p_0 dt = -\rho \int_0^\infty \frac{\partial \Phi}{\partial t} dt \qquad (0.1)$$

is assumed to be finite.

We shall suppose that the body has two mutually perpendicular planes of symmetry perpendicular to the wave front (this symmetry is not only geometrical but also maintained with respect to the mass distribution inside the body). This limitation is imposed only to avoid difficult computations. It will be shown below that the problem can be also solved for a body of a completely arbitrary shape. The weight of the body may be smaller, larger, or equal to the weight of the displaced liquid. We shall, however, neglect displacements of the body due to positive or negative buoyancy.

It is to be proved that with the properties of the pressure wave described above, the displacement of the body tends to some limit (as $t \to \infty$). Beside that, it is required to find that limit.

The problem is solved within the acoustic approximation.

1. Differential equations of the problem. When the above limitations regarding the symmetry of the body are imposed, the body will move gradually in the direction of propagation of the pressure wave, i.e. in the direction of the X-axis.

The differential equation of motion of the body can be written in the following fashion:

$$M \frac{d^2U}{dt^2} = \rho \iint \frac{\partial \Phi}{\partial t} \cos nx dS + \rho \iint \frac{\partial \varphi}{\partial t} \cos nx dS \tag{1.1}$$

where U is the displacement of the body, M is the mass of the body, ρ is the density of the fluid, n is the direction of the exterior normal to the surface of the body $\phi(x, y, z, t)$ is the potential of the diffraction wave.

The integration of (1.1) is performed over the entire surface of the body.

The potential ϕ has to satisfy the three-dimensional wave equation

$$\Delta \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{1.2}$$

with the initial conditions

$$\varphi = \frac{\partial \varphi}{\partial t} = 0 \quad \text{at } t = 0 \tag{1.3}$$

As $r = (x^2 + y^2 + z^2)^{1/2} \to \infty$ the function $\phi \to 0$, and the following condition holds on the surface of the body

$$\frac{\partial \varphi}{\partial n} = -\frac{\partial \Phi}{\partial n} + \frac{dU}{dt} \cos nx \tag{1.4}$$

After integrating (1.1) twice with respect to time (with the limits from $t = 0$ to t), taking into account all available initial data, we obtain

$$MU = \rho \iint \Phi^* \cos nx dS + \rho \iint \varphi^* \cos nx dS \tag{1.5}$$

where

$$\Phi^* = \int_0^t \Phi dt, \qquad \varphi^* = \int_0^t \varphi dt \tag{1.6}$$

The displacements of the particles of the fluid are expressed in terms of these two functions by the formulas

$$\mathbf{v} = \operatorname{grad} \Phi^*, \qquad \mathbf{w} = \operatorname{grad} \varphi^* \tag{1.7}$$

where $\mathbf{v}$ is the displacement caused by the incident wave (i.e. a displacement which would not exist if there were no body in the fluid), and $\mathbf{w}$ is the additional displacement caused by diffraction. Since the incident wave propagates along the X-axis

$$\mathbf{v} = \frac{\partial\Phi^*}{\partial x}\,\mathbf{i}_x = v\mathbf{i}_x \tag{1.8}$$

The function ϕ^* has to satisfy equation

$$\Delta\varphi^* = \frac{1}{c^2}\frac{\partial\varphi}{\partial t} \tag{1.9}$$

and the boundary condition at the surface of the body

$$\frac{\partial\varphi^*}{\partial n} = -\frac{\partial\Phi^*}{\partial n} + U\cos nx = (U - v)\cos nx \tag{1.10}$$

Equations (1.9) and (1.10) are obtained by means of integrating (1.2) and (1.4) with respect to time and taking into account (1.3). Note that if the integral (0.1) is finite (as it was discussed earlier) then also the displacement v will be finite as $t \to \infty$, and will tend to some limit

$$v_\infty = \lim_{t\to\infty}\frac{\partial\Phi^*}{\partial x} \tag{1.11}$$

The potential of the incident wave Φ (and consequently also its integral Φ^*) has no singularities inside the region occupied by the body. One can write on this basis

$$\rho\iint\Phi^*\cos nx\,dS = \rho\iiint\frac{\partial\Phi^*}{\partial x}\,dV = \rho\iiint v\,dV \tag{1.12}$$

where the integration on the right is performed over the entire volume occupied by the body. The other integral of (1.5) can be written according to (1.10) in the following form:

$$\rho\iint\varphi^*\cos nx\,dS = \rho\iint\frac{1}{U-v}\,\varphi^*\frac{\partial\varphi^*}{\partial n}\,dS \tag{1.13}$$

When (1.12) and (1.13) are taken into account formula (1.5) becomes:

$$MU = \rho\iiint v\,dV + \rho\iint\frac{1}{U-v}\,\varphi^*\frac{\partial\varphi^*}{\partial n}\,dS \tag{1.14}$$

2. Solution of the problem. In order to find $U(t)$ it is necessary to know the function $\phi^*(x, y, z, t)$, which is, of course, impossible with the above general statement of the problem.

Therefore, we shall not seek $U(t)$, but only the final displacement of the body

$$U_\infty = \lim_{t\to\infty} U(t) \tag{2.1}$$

Note that, in general, it is possible that such a limit may not exist. Thus, for instance, if the wave had the form of a pressure jump, then because of its action, the body would acquire some constant velocity [1]. If, however, the total pressure impulse (0.1) is limited, then the fluid particles will acquire finite displacements and one can expect that in this case also the displacement of the body will be finite. Let us assume that this is the case and let us see where this assumption will lead us.

Thus, let for $t \to \infty$, $v \to v_\infty$, $U \to U_\infty$, and then it follows from equation (1.14) that

$$MU_\infty = M_0 v_\infty + \frac{\rho}{U_\infty - v_\infty} \lim_{t\to\infty} \left\{ \iint \varphi^* \frac{\partial \varphi^*}{\partial n} dS \right\} \qquad (2.2)$$

where M_0 is the mass of the liquid displaced by the body. Thus, it is necessary to determine

$$\lim_{t\to\infty} \left\{ \iint \varphi^* \frac{\partial \varphi^*}{\partial n} dS \right\} = \iint \varphi_\infty{}^* \frac{\partial \varphi_\infty{}^*}{\partial n} dS \qquad (2.3)$$

where

$$\varphi_\infty{}^* = \lim_{t\to\infty} \varphi^* = f(x, y, z) \qquad (2.4)$$

The function ϕ^* has to satisfy equation (1.9), whose right-hand side approaches zero as $t \to \infty$ (since it is proportional to the pressure in the diffraction wave). Thus $\phi_\infty{}^*$ is a harmonic function. It damps out as $r \to \infty$, and on the surface of the body it is subject to the condition

$$\frac{\partial \varphi^*{}_\infty}{\partial n} = A \cos nx \qquad (A = U_\infty - v_\infty = \text{const}) \qquad (2.5)$$

From this it follows that $\phi_\infty{}^*$ can be identified with the flow potential of an infinite ideal fluid with the body under study moving in it at a constant velocity A in the direction of the X-axis. Besides, we are not interested in the function itself but only in the integral (2.3).

We transform it by Green's formula keeping in mind that as $r = (x^2 + y^2 + z^2)^{1/2} \to \infty$ the function $\phi_\infty{}^*$ tends to zero as r^{-2} ([2] p. 370).

Then we obtain

$$\iint \varphi_\infty{}^* \frac{\partial \varphi_\infty{}^*}{\partial n} dS = \frac{1}{2} \iint \frac{\partial (\varphi_\infty{}^*)^2}{\partial n} dS = -\iiint \left[\left(\frac{\partial \varphi_\infty{}^*}{\partial x} \right)^2 + \left(\frac{\partial \varphi_\infty{}^*}{\partial y} \right)^2 + \left(\frac{\partial \varphi_\infty{}^*}{\partial z} \right)^2 \right] dV \qquad (2.6)$$

At the right-hand side of this equation the integration is performed over the region where $\phi_\infty{}^*$ is given, i.e. over the entire volume occupied by the fluid surrounding the body.

Thus the problem is reduced to the evaluation of the integral

$$T = \frac{1}{2} \rho \iiint \left[\left(\frac{\partial \varphi_\infty{}^*}{\partial x} \right)^2 + \left(\frac{\partial \varphi_\infty{}^*}{\partial y} \right)^2 + \left(\frac{\partial \varphi_\infty{}^*}{\partial z} \right)^2 \right] dV \qquad (2.7)$$

This is nothing else, however, but the kinetic energy of the ideal incompressible fluid in a problem with the boundary condition (2.5). From this we can write immediately ([2] p. 379, 384)

$$T = \frac{1}{2} M_0 \mu_x A^2 \tag{2.8}$$

where μ_x is the coefficient of additional mass for the body studied (moving in the direction of the X-axis).

On the basis of (2.6)-(2.8) and (2.5), expression (2.1) becomes

$$MU_\infty = M_0 v_\infty - M_0 \mu_x A = M_0 v_\infty - M_0 \mu_x (U_\infty - v_\infty) \tag{2.9}$$

After solving this equation for U_∞ we obtain

$$U_\infty = (1 + \mu_x) \frac{v_\infty}{\mu_x + M / M_0} \tag{2.10}$$

Thus, the assumption that there exists a limiting value of the displacement (2.1) did not lead us to any contradiction and is confirmed by the final formula (2.10).

The problem studied here can be also easily solved for an absolutely rigid body of a completely arbitrary shape. Such a body will experience a motion along all three coordinate axes due to the passing of an acoustic pressure wave with a finite impulse. In addition, it will rotate by some angles around the axes. By means of a reasoning similar to the foregoing one, using the information given in [2], p. 368, one can derive for the six unknown quantities (three displacement components and three rotation components) a linear algebraic system. The coefficients of this system will depend on the 21st coefficient of additional masses and on the static moments. We shall not go into this, however, in greater detail, since the particular case studied above is of greatest interest, and since it sufficiently discloses the method of derivation also for the general case.

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